IJESRT

INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & RESEARCH TECHNOLOGY

A STUDY ON THE 3-DIM. EIGENMODE OF THE RECTANGULAR RESONANT CAVITY BY FEM

Yeong Min Kim*
* Dept. of Elecronic Physics, Kyonggi University, Korea

ABSTRACT
FEM has been applied to obtain the 3-dim. eigenmode of the rectangular resonant cavity. The vector Helmholtz equation has been analysed for the resonant field strength in homogeneous media. An eigen-equation has been constructed from element equations basing on tangential edges of the tetrahedra element. This equation made up of two square matrices associated with the curl-curl form of the Helmholtz equation. To obtaine the more stable result, the equation was treated with the shift-invert strategy. By performing Krylov-Schur iteration loop on the eigen-equation, the matrix has been transformed into the Schur form. Eigen-values have been determined from diagonal elements of the Schur matrix. Eigen-modes were determined from the unitary similar transforming matrix of Krylov-Schur iteration loop. Eigen-pairs as a result have been revealed visually in the schematic representations.

KEYWORDS: FEM, Helmhotz, tangential, tetrahedral, eigenmode, Krylov-Schur, rectangular, cavity.

INTRODUCTION
Usually, physical property of the resonant cavity would be determined depending on its shape. Shape of the resonant cavity influence the characteristics of the electromagnetic field in it. Shape of the resonant cavity is to be designed according to its applying purpose. Eigenmode represents a significant feature of the resonant cavity. Knowledge about eigenmode is one of the most important thing in designing the resonant cavity. Acquiring information about the eigen property is indispensable to make more advanced products. In most cases, infinite numbers of the discret eigen pair are formed in the resonant cavity. The theoretical analysis of the eigenmode is possible. However, solving the equation involves a difficulty. Especially, the 3-dim.(dimensional) resonant cavity make the complex eigen-pairs which can not be easily identified even in the simple geometric strucutre. So, the numerical analysis is required to understand the properties of the specific eigenmode. It would be resonable to study various eigen-pairs for the resonant cavity using the more confidential numerical algorithm as like FEM(Finite Element method). Based on FEM, the matrix eigen equation can be established from the vector Helmholtz equation. For a three-dimensional problem, the number of variables increases drastically compared with those for a two-dimensional problem. Hence it is not economical to use a generalized eigenvalue solver. Krylov-Schur iteration method has been known as one of the most important and actively developing algorithms for calculating the huge dimensional eigen-problems[1][2]. Previously, we have studied on the eigen-properties of 2-dim. waveguides of various forms using Krylov-Schur iteration method[3][4]. And the eigen-pairs of the 3-dim. cylindrical resonant cavity were also obtained using the same algorism[5]. From these studies, it could be recognized ones again the prominent ability of Krylov-Schur algorithm in calculating the large scale and non-symmetric eigen-problems. In this study, Krylov-Schur algorithm the same as previously studying has been applied to a 3-dim. resonant cavity of the rectangular shape. The eigen-equation were constructed basing on FEM. The mesh element was simple tetrahedron and the shape functions were constructed with constant tangential edge vectors. To obtaine the more stable result, the equation was treated with the shift-invert strategy. The Krylov-Schur algorithm was carried out on this equation to obtaine the eigen mode characterizing the wave properties of the cavity. As the results, the spectra for each eigen-pairs have been visualized with the schematic representations as like the previous study.

FINITE ELEMENT FORMULATION
The following description for calculating the eigen-modes is the same as describing in reference [5]. The formulation can be followed by using either the $\vec{E}$ or $\vec{B}$ feld. For a convenience of calculation, only $\vec{E}$ would be discussed. The

vector Helmholtz equation would be used in determining the wave property of the resonant cavity. It is described as following equation [6] [7]

$$\vec{\nabla} \times \left(\frac{1}{\mu_r} \nabla \times \vec{E} \right) - k^2 \varepsilon_r \vec{E} = 0 \qquad (1)$$

Where k is the wave number and, for $\vec{E}$(electric field strength), μ_r(relative permeability μ/μ_o), ε_r(relative permittivity $\varepsilon/\varepsilon_o$). The eigen-equation is constructed from FEM basing on the tetrahedral elemental mesh. The rectangular resonant cavity and the tetrahedral mesh is shown in the Fig.1. In the calculation, boundaries of the cavity have been assumed to be PEC(perfect electric conductor). Hence, for the TM and normal derivative for the TE cases may vanish at the boundary.

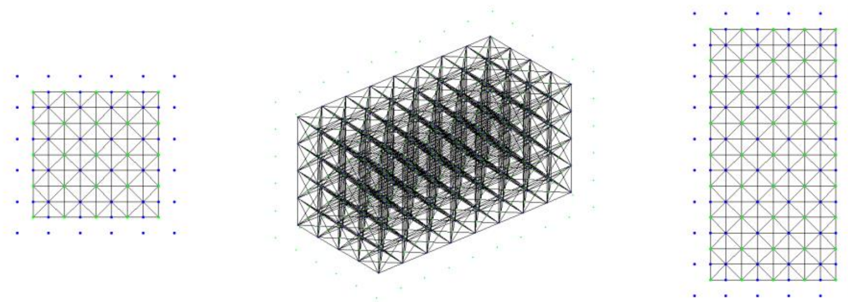

Figure 1 The rectangular cavity and tetrahedral mesh.

The Galerkin method of weighted residual has been used to construct a linear equation[6][7][8]. The equation resulting for this method is given as following

$$\iiint \frac{1}{\mu_r} (\vec{\nabla} \times \vec{T}) \cdot (\vec{\nabla} \times \vec{E}) dV = k_o^2 \varepsilon_r \iiint \vec{T} \cdot \vec{E} \mathrm{dV} \qquad (2)$$

where $\vec{T}$ is a weighting function. To avoid the spurious solution attributed to the lack of enforcement of divergence condition for $\vec{E}$, basis functions have been constructed with constant tangential edge vectors $\vec{W}_m$of the tetrahedral element

$$\vec{W}_m = l_m \left(N_{m1} \vec{\nabla} N_{m2} - N_{m2} \vec{\nabla} N_{m1} \right), \quad m = 1, 2, 3, 4, 5, 6. \qquad (3)$$

In this representation, $N_{m1}$and N_{m2} are the simplex coordinates associated with the 1st and 2nd nodes connected by the edge m, and l_m is the length of edge m. The simplex coordinates for a given elementary mesh are

$$N_n = a_n + b_n x + c_n y + d_n \mathrm{z}, \quad \mathrm{n} = 1, 2, 3, 4 \qquad (4)$$

And the gradient of any coordinate is

$$\vec{\nabla} N_n = b_n \hat{x} + c_n \hat{y} + d_n \hat{z} \qquad (5)$$

The simplex coefficients are calculated by inverting the coordinate matrix

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}^{-1} \qquad (6)$$

Where (x_n, y_n, z_n) is a rectangular coordinate of the node n of the tetrahedral mesh. For each elemental mesh, edges and nodes are related with each other as illustrated in Fig.2.

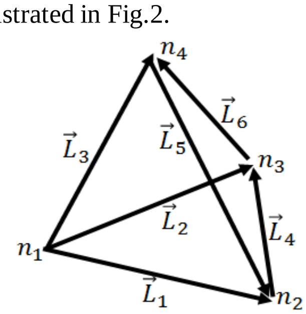

Figure 2 The tetrahedral element mesh

The electric field strength in a single tetrahedral element is represented with the tangential edge vector as

$$\vec{E} = \sum_{m=1}^{m=6} e_m \vec{W}_m \quad (7)$$

The six unkown parameters $e_1, \ldots, e_6$ are associate with tangential edges of the tetrahedral elemental mesh. Substituting equation (7) into equation (2), the eigen-equation of one tetrahedral element can be written in matrix form as

$$[S_{el}][e] = k^2[T_{el}][e] \quad (8)$$

where the element matrics are given by

$$[S_{el}] = \iiint \frac{1}{\mu_r}(\vec{\nabla} \times \vec{W}) \cdot (\vec{\nabla} \times \vec{W}) dV \quad (9)$$

$$[T_{el}] = \varepsilon_r \iiint \vec{W} \cdot \vec{W} \mathrm{dV} \quad (10)$$

The evaluation of the element matrix requires the curl of each basis function $\vec{W}_m$

$$\begin{aligned} \vec{\nabla} \times \vec{W}_m &= \vec{\nabla} \times l_m (N_{m1} \nabla N_{m2} - N_{m2} \nabla N_{m1}) \\ &= 2 l_m \vec{\nabla} N_{m1} \times \vec{\nabla} N_{m2} \\ &= 2 l_m \big((c_{m1} d_{m2} - c_{m2} d_{m1}) \hat{x} + (b_{m2} d_{m1} - b_{m1} d_{m2}) \hat{y} + (b_{m1} c_{m2} - b_{m2} c_{m1}) \hat{z} \big) \\ &\equiv 2 l_m \vec{w}_m \end{aligned} \quad (11)$$

And from itFrom the equation

$$[S_{el}]_{mn} = 4 l_m l_n V (\vec{w}_m \cdot \vec{w}_n) \quad (12)$$

To obtain the element matrix $[T_{el}]$, the scalar product between $\vec{W}_m$ and $\vec{W}_n$ may be calculated as

$$\vec{W}_m \cdot \vec{W}_n = l_m (N_{m1} \vec{\nabla} N_{m2} - N_{m2} \vec{\nabla} N_{m1}) \cdot l_n (N_{n1} \vec{\nabla} N_{n2} - N_{n2} \vec{\nabla} N_{n1}) \quad (13)$$

$$= l_m l_n [N_{m1} N_{n1} \varphi_{m2,n2} - N_{m1} N_{n2} \varphi_{m2,n1} - N_{m2} N_{n1} \varphi_{m1,n2} + N_{m2} N_{n2} \varphi_{m1,n1}] \quad (14)$$

Wher $\varphi_{mi,nj} = \vec{\nabla} N_{mi} \cdot \vec{\nabla} N_{nj} = b_{mi} b_{nj} + c_{mi} c_{nj} + d_{mi} d_{nj}$

In the process of $[T_{el}]$ calculation, following volume intefration for 3-Dim. Simplex coordinates may be used[9]

$$\iiint (N_1)^i (N_2)^j (N_3)^k (N_4)^l dV = \frac{3!\, i!\, j!\, k!\, l!}{(3 + i + j + k + l)!} V \quad (15)$$

These integrals can be simply summarized in the following matrix form

$$[M_{ij}] = \frac{1}{V} \iiint N_i N_j dV = \frac{1}{20} \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \quad (16)$$

From the equations (13), (14) and (16), the element matrix can be written as following

$$[T_{el}]_{mn} = V l_m l_n [\varphi_{m2,n2} M_{m1,n1} - \varphi_{m2,n1 M_{m1,n2}} - \varphi_{m1,n2 M_{m2,n1}} + \varphi_{m1,n1} M_{m2,n2}] \quad (17)$$

These element matrices are assembled over all tetrahedral elements in the 3-Dim. cavity to obtain a global eigen-matrix equation.

$$[S][e] = k^2[T][e] \quad (18)$$

RESULTS AND DISCUSSION

In this study, the eigen-pairs of the rectangular resonant cavities have been investigated with Krylov-Schur iteration method. For a convenience of the calculation, the lateral surface of the cavity was assumed to be coated with the perfectly conducting metal. The space occupied by the cavity was supposed to be linear and homogeneous. So, it has not been worried any leakage and anisotropic field variation in the calculation. For FEM calculation, the mesh was constructed with the tetrahedral structure as can be seen in Fig.1. The tangential edge vectors were related with the vertics of all elemental tetrahedrons as illustrated in Fig.2. Here, the definition of the tetrahedral volume follow the right handed spiral rule. If the vertices of a tetrahedron following the right hand rule, the volume is for positive value and reverse for negative. The eigen-equation of the matrix form eq.(18) was established basing on the these tetrahedrons. As mentioned in the previous study, it has been well known that the Krylov-Schur iteration method is the most reliable technique for finding the prominent eigen-modes. The method would be more efficiently implemented in finding specific eigen-pairs by performing the shift-invert strategy as following[10]

$$\lambda[\mathrm{e}] = \frac{[T]}{[S] - \sigma[T]}[e] = [M][e] \quad (19)$$

where $\lambda = \frac{1}{k^2 - \sigma}$. In this study, the relative permeability and relative permittivity was assumed to be $\mu_r = 1$ and $\varepsilon_r = 1$ respectively for a convenience.The sparsity and symmetry of the eigen-equation would be lost, but by this strategy the convergent rate may be promoted at the specific value σ. The Krylov-Schur iteration method has been performed on this square matrix [M]. By this iteration method, the matrix [M] has been transformed into a Schur matrix. The eigen-modes are the column vectors of the similar transforming matrix which convert the square matrix $[M]$ to a Shure form. The eigen-values are calculated by converting each diagonal component of the Schur matrix into values $k^2 = \frac{1}{\lambda} - \sigma$ reversing the shift-invert strategy. As a result, the eigen-pairs are schematicaly represented in the fig. 3. The wave numbers calculated from shift-invers relation were written in the blanket under each spectrum. As can be seen in the spectra, each eigenmode is shown the complicated configuration. However, each of the spectra show the uniform electric field strength oriented to a specific direction. The specific mode type may be determined readily by investigating the direction of the electric field strength. These mode type are shown under each spectrum accompanying with a eigenvalues value.

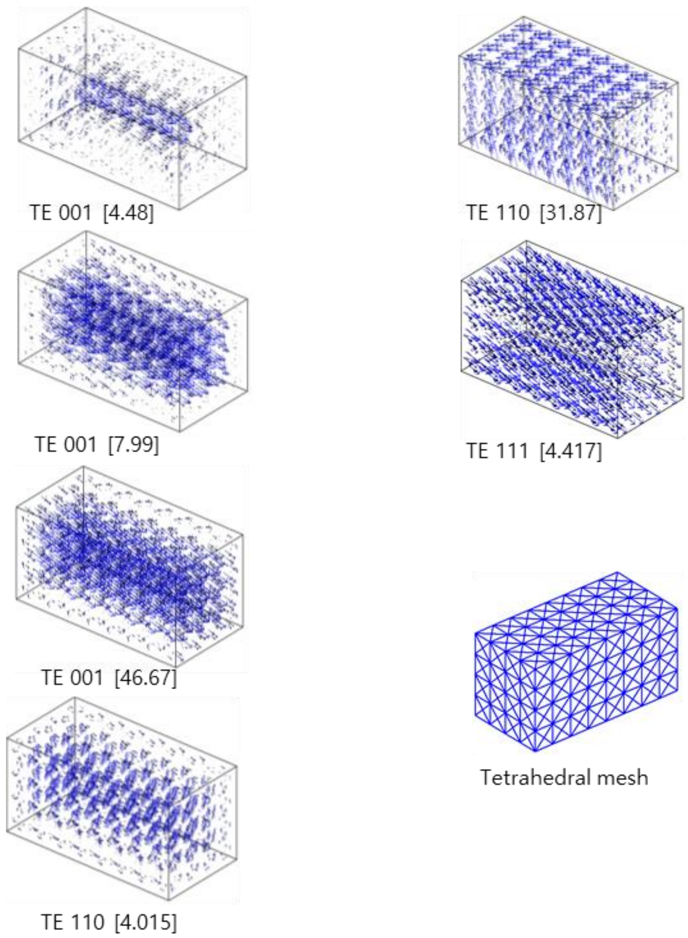

Figure 3 Schematic representation of 3-Dim. eigen-pairs

CONCLUSION

The 3-dim. eigen-equation of the rectangular resonant cavity has been constructed by FEM. Eigen-pairs have been calculated by applying the Krylov-Schur iteration method to the shift-invert matrix. As a result, the shift-invert matrix was transformed in the Schur matrix. The wave numbers were determined by reversing the shift-invert strategy for the diagonal elements of the Schur matrix. The eigen-modes were obtained from the column vectors of the similar transforming matrix which convert the eigen-equation to a Shure form.

REFERENCES

[1] G. W. Stewart, "A Kryliv Schur Algorithm for Large Eigenproblems" SIAM J. Matrix Anal. & Appl. **23(3)**,

601 (2002).
[2] G. W. Stewart, "Addendum to 'A Krylov-Schur Algorism for Large Eigenproblems' " SIAM J. Matrix Anal. & Appl. 24(2), 599 (2002).
[3] Yeong Min Kim, Dongchool Kim and Jong Soo Lim, "A Study on The Eigen-properties of A 2-D Square Waveguide by the Krylov-Schur iteration method" Journal of The Institute of Electronics Engineers of Korea Vol. 50. NO. 11, November 2013.
[4] Yeong Min Kim, and Jong Soo Lim, "A Study on The eigen-properties on Vaired structural 2-Dim. Waveguides by Krylov-Schure Iteration Method" Journal of The Institute of Electronics Engineers of Korea Vol. 51. NO. 2, Faburary 2014.
[5] Yeong Min Kim and Jong Soo Lim, "A study on eigen-properties of a 3-Dim. resonant cavity by Krylov-Schur iteration method" Journal of The Institute of Electronics Engineers of Korea Vol. 51. NO. 7, July 2014.
[6] C. J. Reddy, Manohar D. Deshpande, C. R. Cockrell, and Fred B. Beck, "Finite Element Method for Eigenvalue Problems in Electromagnetics" NASA Technical Paper 3485(1994).
[7] Alexandre Nentchev, "Numerical Analysis and Simulation in Microelectronics by Vector Finite Elements" Ph. D. thesis, Vienna University of technology, 2008.
[8] J. Scott Savage, and Andrew F. Peterson, "Higher-Order Vector Finite Elements for Tetrahedral Cells" IEEE Trans. Microwave theory Tech. Vol. 44, No. 6, June 1996
[9] P. Silvester and R. Ferrari, *Finite Elements for Electrical Engineering*, 3th ed.(Cambridge University Press, New York,1996), p. 183.
[10] Maysum Panju "Iterative Methods for Computing Eigenvalues and Eigenvectors" University of Waterloo, http://mathreview.uwaterloo.ca

AUTHOR BIBLIOGRAPHY

Yeong Min Kim
He received B.S. degree in department of physics from Kyonggi University, Korea in 1985. He completed the M.S. and Ph.D. degrees with the Solide State Physics from DongGug University, Korea in 1987 and 1995 respectively. He is working now as a Visiting Professor of department of Electronic Physics in Kyonggi University, Korea since 2013. His special interesting field was photochromic using Mössbauer spectroscopy